Barrier Option Pricing With Cash Market Closing

Luo Long

Mathematical Institute, Andrew Wiles Building, Woodstock Road, Oxford, OX2 6GG, UK

Email: luo.long@stx.ox.ac.uk

Manuscript received May 30, 2023; revised January 16, 2024; accepted January 29, 2024; published February 3, 2024

***Abstract*—The vanilla European barrier options on stocks with the condition that the market closes and opens periodically in a day are given analysis. Inspired by Vitalis Siven et al., *Barrier options and lumpy dividends* 2009, we show that in the Black-Scholes model under the principal risk-neutral analysis with the cash market close, dividend-free barrier option prices can be expressed in terms of well-behaved one- or multi-dimensional integrals, depending on the number of simulation days. With more than one day, the higher integral dimensions cause more computational power and longer time, but we show that especially for the down-and-out barrier options, the option prices for longer simulation periods can be approximated directly.**

***Keywords*—barrier option, market close, numerical integration, down-and-out call**

I. Introduction

The close of the underlying stock market affects stock price dynamics and consequently the pricing of equity derivatives. After the market has closed, the invisibility of the stock price paths comes into existence. For example, if the price path touches the barrier level during the closing time of the market, we can not assume it is absorbed since we can not observe the price paths like what we do when the market is open. Due to the invisibility of the price path when the market is down, the option type turns into a plain-vanilla option since the price path follows the geometric Brownian motion (GBM) and varies without restrictions until the next market open. The ramifications of the market close are not significant if we are considering plain-vanilla options in the day. However, the issue becomes more acute in the case of barrier options, not only because the close of the underlying stock market forces the option type to be plain-vanilla, eliminating the possibilities of barrier crossing at night, but also affects the final pricing of this special kind of barrier options involving partial vanilla option characteristics during the whole maturity time (see Fig. 1). To facilitate terms and descriptions, we refer this special kind of barrier option as 'BV option'.

This literature contains a systematic approach to solving the pricing issue considering the cash market close with consistent comparisons with the simulations results utilizing Python scripts to guarantee the accuracy of the formulas and the eventual values. In Section II, we review some foundations for pricing European options with no dividend payments [1–3].

In Section III, we further the method illustrated in Vitalis, Suchaneck *et al*., [4] and integrate it to price the BV option with a time maturity of one day. The important point is that under no dividend payments and given that the option has not been knocked out in the day, the barrier option turns into a vanilla option at night to eliminate the knock-out possibilities. Due to the fact that barrier options are weakly path-dependent, the price is just a double-variable function of time and the current stock price. In the dividend-free Black-Scholes model, the barrier option price is a one-dimensional integral of Black-Scholes-type functions underpinned by a log-normal density. With adjusted integration limits and contract pricing functions, the integral can be evaluated numerically.

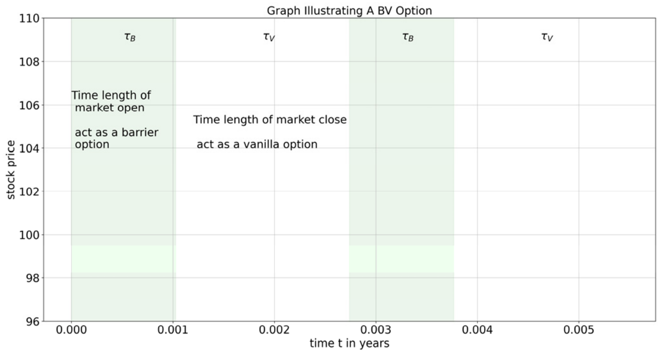

Fig. 1. Graphical illustration of BV options. Here, it is a two-day BV option.

Then, we move to demonstrate the deduced numerical integration method can be modified to price the BV option within two days, i.e., the market closes two times. The following Section IV contains simulation results to illustrate the accuracy of the results upon the applications of the integration formulas deduced in the previous sections and we will present an approximation approach to obtain the prices of the BV options when the maturity time is three days or above.

II. European Options on Non-dividend-paying Stock

This section reviews some approaches for pricing European-type contracts underpinned by a non-dividend-paying stock. In the presence of dividend payments, please consult [4] and [5].

Consider a time interval $[0, T]$ in which we examine the stock price S at a deterministic time $\tau \in (0, T)$, i.e., the market closing time. Let $r \geq 0$ denote the risk-free interest rate and $S_{t\in[0,T]}$ be the stock price at time t. The application of the geometric Brownian motion, or GBM, describes the stock price under the risk-neutral measure. During the time interval $[0, T]$, the stock price evolution obeys GBM property, i.e., the usual Black-Scholes model, except at the examination time τ, where the stock price can be manually affected such as the change in the option type. Hence, we have:

$$dS_t = rS_t\, dt \;+\; \sigma S_t\, dW_t, \qquad t \in [0, \tau) \cup (\tau, T] \tag{1}$$

$$S_\tau = S_{\tau^-} \tag{2}$$

here, $W_{t\in[0,T]}$ is a standard Brownian motion in the Black-Scholes model. Meanwhile, immediately before the time τ, denoted as τ^-, S_τ depends on how the S_{τ^-} is affected. For example, if a dividend is included, Eq. (2) can be modified as $S_\tau = S_{\tau^-} - d$, where d is the dividend, described in [4] and [5] . Meanwhile, from risk-neutral valuation, the arbitrage-free price $F(t, S_t)$ of a European option can be described as

$$F(t, S_t) = e^{-r(T-t)}\mathbf{E}\left[\Phi\left(S(T)\right)\right] \tag{3}$$

Eq. (3) denotes the price at time t of a European option with the payoff function Φ and maturity time $T - t$. For example, for a standard vanilla call option with strike price K and maturity T, $\Phi(S(T)) = (S_T\text{-}K)^+$. Here, t describes the current time, S_t for the current stock price, and $E\left[\Phi\left(S(T)\right)\right]$ is the expectation value of the final payment starting from the time t and evolving for $T - t$. Suppose we focus on the price of a European call option at time t with strike K and expiry T given that $S_t = s$, using Eq. (3), we shall obtain $C(t, s) = e^{-r(T-t)}\mathbf{E}[(S_T - K)^+ | S_t = s]$. Referring to Eq. (1) and Eq. (2), after picking up a time τ for examination purposes, it is noticed that the stock price follows GBM on $[\tau, T]$ — the price $C(\tau, s)$ is, therefore, the common Black-Scholes price. To compute the contract price at time 0, we simply do the following modifications:

$$\begin{aligned} e^{-rT}\mathbf{E}[(S_T - K)^+] &= e^{-r\tau}\mathbf{E}\left[e^{-r(T-\tau)}E[(S_T - K)^+, S_\tau]\right] \\ &= e^{-r\tau}\mathbf{E}[C(\tau, S_\tau)] \\ &= e^{-r\tau}\mathbf{E}[C(\tau, S_{\tau^-})] \end{aligned} \tag{4}$$

Thus, we can obtain the price of the European option with strike K and maturity T at time 0 by examining an arbitrary time $\tau \in [0, T]$ and considering the expectation value of its price at time τ with the corresponding stock price S_{τ^-}. In fact, the last expression is a single integral of the pricing function $C(\tau, s)$ multiplied by a log-normal density suggested by the Black-Scholes model.

III. Pricing One-day, Two-day BV Options and the Generalization

Now that we can price a European option by constructing integrals from a selected time $\tau \in [0, T]$, we extend this approach to treat BV options with a maturity time of one day. For simplicity, we focus on a down-and-out call option only, but similar approaches can be easily seen for other contracts.

We compute the price at time 0 of a BV option with initial price S_0, expiry T, strike price K, volatility σ, barrier L, with $0 \le L \le K$. The corresponding T-contract Z is written on a non-dividend-paying stock as

$$Z = \begin{cases} \Phi\left(S(T)\right) & \text{for all } t \in (\tau, T] \\ \Phi\left(S(\tau)\right) & \text{if } S(t) > L \text{ for all } t \in [0, \tau] \\ 0 & \text{if } S(t) \le L \text{ for some } t \in [0, \tau] \end{cases} \tag{5}$$

The time-T payoff is $(S_T - K)^+$. Meanwhile, we note that for $[0, \tau]$, the option is a down-and-out barrier option, or an absorbing process, with a time period τ. Applying Eq. (4), we show that the price $P(t, s)$ for such a BV option at time 0 is, by considering the absorption process in $[0, \tau]$,

$$P(0, S_0) = e^{-r\tau}\mathbf{E}[C(\tau, S_\tau)] = e^{-r\tau}\int_{\ln L}^{\infty} C(\tau, S_\tau) f(x)dx, \tag{6}$$

where $C(t, s)$ denotes the usual Black-Scholes price formulas and $f(x)$ denotes the density distribution of the absorbed process. Inspired by [6] and [7], from [8] (Proposition 18.3), we have the density distribution as

$$\begin{aligned} f(x) &= \varphi\left(x; \tilde{r}\tau + \ln S_0, \sigma\sqrt{\tau}\right) - \exp\left[-\frac{2\tilde{r}(\ln S_0 - \ln L)}{\sigma^2}\right]\varphi\left(x; \tilde{r}\tau - \ln S_0 + 2\ln L, \sigma\sqrt{\tau}\right) \\ &= \varphi\left(x; \tilde{r}\tau + \ln S_0, \sigma\sqrt{\tau}\right) - \left(\frac{L}{S_0}\right)^{2\tilde{r}/\sigma^2}\varphi\left(x; \tilde{r}\tau + \ln\frac{L^2}{S_0}, \sigma\sqrt{\tau}\right) \end{aligned} \tag{7}$$

where $\tilde{r} = r - \frac{1}{2}\sigma^2$, $\varphi(x; m, v)$ as a normal distribution about x with mean m and standard deviation v. Hence, by substituting Eq. (7) into Eq. (6) and with appropriate transformations of variables, we show that $P(0, S_0)$ can be illustrated as

$$P(0, S_0) = e^{-r\tau}\left(I(S_0) - \left(\frac{L}{S_0}\right)^{2\tilde{r}/\sigma^2} I\left(\frac{L^2}{S_0}\right)\right) \tag{8),}$$

where

$$I(s) = \int_{l(s)}^{\infty} C\left(\tau, se^{\left(r - \frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}z}\right)N'(z)dz, \qquad l(s) = \frac{\ln\frac{L}{s} - \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} \tag{9}$$

here, $N'(\cdot)$ denotes the density of the standard normal distribution. The Eq.(8) corroborates [8] (Theorem 18.8). The integral in Eq.(9) can be done using numerical computation since the integrand is smooth and converges due to the factor $N'(\cdot)$. In the case of a pure vanilla call option (maturity T, strike K, initial stock price S_0), by letting $L = 0$, we have the corresponding integral expression for the price at time 0, $V(0, S_0)$:

$$V(0, S_0) = e^{-r\tau}\int_{-\infty}^{\infty} C\left(\tau, se^{\left(r - \frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}z}\right)N'(y)dy \tag{10}$$

In the following part, we extend Eqs. (8-9) to derive the pricing formula for two-day BV options.

First, we postulate that for the second day, the stock price will evolve under the same combination of options as the first day, i.e., the barrier option and the vanilla option, from which we can assume the price path undergoes a periodic cycle though with changing stock price at each time steps. Compared to the one-day case, we aim to hold all other parameters as constants such as volatility and market open/close time length, thus its option price only depends on the initial stock price S_0. Hence, we speculate that S_0 can be treated as a variable. Thus, if we can know the expression for the starting initial price on the second day, we can substitute it as S_0 in the Eq. (8). Hence, generally, we adopt this treatment for the last day of the N-day BV option with S_0 as the starting stock price of the last day before the option reaches its maturity time. Thus, to price this N-day BV option, all we are left to deduce is the expression of the starting stock price of the last day, i.e., the N^{th} day. In this two-day case, we adopt what has been proposed and re-denote S_0 as s:

$$P(s) = e^{-r\tau}\left(I(s) - \left(\frac{L}{s}\right)^{2\tilde{r}/\sigma^2} I\left(\frac{L^2}{s}\right)\right) \tag{11}$$

where $\tilde{r} = r - \frac{1}{2}\sigma^2$. Hence, if we can figure out the expression of s, i.e., the expression of the stock price after the initial stock price S_0 has evolved through the market open period (the barrier-option period) and the market close period (the vanilla option period), the pricing formula for the two-day BV option can be obtained. To achieve it, we exploit the GBM property of the stock price. From GBM, we have the distribution of $\ln S_T - \ln S_0$ as $\varphi\left[\left(r-\frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right]$, where $\varphi(x; m, v)$ as a normal distribution about x with mean m and standard deviation v.

$$\ln\left(\frac{S_T}{S_0}\right) = \left(r - \frac{\sigma^2}{2}\right)T + \sigma\epsilon\sqrt{T} \tag{12}$$

where ϵ is a value drawn from a standard normal distribution. Rearranging gives:

$$S_T = S_0\, e^{\left(r-\frac{\sigma^2}{2}\right)T + \sigma\epsilon\sqrt{T}} \tag{13}$$

Meanwhile, the existing condition of the starting price of the last day is that the previous price path has not been absorbed, implying that for the previous time period, we can safely adopt Eq. (13) to describe how the stock price has evolved. Based on the speculations, we reach the expression for the last-day initial stock price s:

$$s(S_0, x, y) = S_0\left[e^{-\frac{(r-\sigma^2)}{2}\tau_B + \sigma\sqrt{\tau_B}y}\right]\left[e^{-\frac{(r-\sigma^2)}{2}\tau_V + \sigma\sqrt{\tau_V}x}\right] \tag{14}$$

where τ_B and τ_V denote how long the market remains open in a day and how long the market remains closed in a day respectively; y and x are the random variables in standard normal distributions related to the barrier option period and the vanilla option period respectively; S_0 denotes the initial stock price at the beginning. Fig. 2 illustrates the meaning of τ_B and τ_V. Thus, the pricing formula for two-day BV options is, after substituting Eq. (14) into Eq. (11),

$$P_2(S_0) = e^{-r\tau}\left(I_2(S_0) - \left(\frac{L}{S_0}\right)^{2\tilde{r}/\sigma^2} I_2\left(\frac{L^2}{S_0}\right)\right); \tag{15}$$

But $I_2(S_0)$ involves a double integral of x and y

$$I_2(S_0) = \int_{\infty}^{\infty}\int_{l(S_0)}^{\infty} C\left(\tau_B, s(S_0, x, y)\right)N'(y)N'(x)dydx; \tag{16}$$

and the lower limit is

$$l_2(S_0) = \frac{\ln\frac{L}{S_0} - \left(r - \frac{\sigma^2}{2}\right)\tau_B}{\sigma\sqrt{\tau_B}} \tag{17}$$

here, x goes from $-\infty$ to ∞ and y goes from $l_2(S_0)$ to ∞. In other words, the pricing formula for two-day BV options is exactly replacing Eq. (9) with Eq. (16)-(17) with the increased dimension in the integral based on Eq. (14). Revisiting the Eq. (14), one can easily see that it consists of three parts: the initial stock price S_0, the GBM process for the stock price under the barrier option when the market is open, and the GBM process under the vanilla option when the market is closed, which forms a cycle of how the stock price changes through a complete day. Hence, generalising this approach, we can revise Eq. (14) for the starting initial stock price $s_N(S_0, x_1, y_1, x_2, y_2, \cdots, x_{N-1}, y_{N-1})$ for the last day in the N-day BV option:

$$\begin{aligned} s_N(S_0, x_1, y_1, x_2, y_2, \cdots, x_{N-1}, y_{N-1}) = S_0 &\left[e^{-\frac{(r-\sigma^2)}{2}\tau_B + \sigma\sqrt{\tau_B}y_1}\right]\left[e^{-\frac{(r-\sigma^2)}{2}\tau_V + \sigma\sqrt{\tau_V}x_1}\right] \times \\ &\left[e^{-\frac{(r-\sigma^2)}{2}\tau_B + \sigma\sqrt{\tau_B}y_2}\right]\left[e^{-\frac{(r-\sigma^2)}{2}\tau_V + \sigma\sqrt{\tau_V}x_2}\right]\cdots \\ &\left[e^{-\frac{(r-\sigma^2)}{2}\tau_B + \sigma\sqrt{\tau_B}y_{N-1}}\right]\left[e^{-\frac{(r-\sigma^2)}{2}\tau_V + \sigma\sqrt{\tau_V}x_{N-1}}\right] \end{aligned} \tag{18}$$

here, $x_i, y_i, i = 1,2,\cdots,N-1$ represent the random variables in standard normal distributions related to the barrier option period and the vanilla option period in each day. By analogy to the previous approach, one can follow similar substitutions and obtain the corresponding N-day BV option pricing formula as indicated by the corresponding N-dimensional integral.

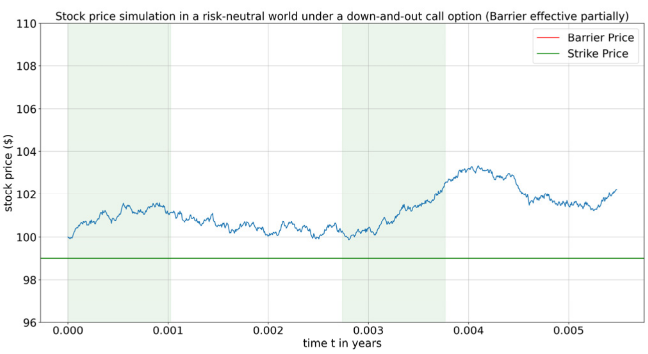

Fig. 2. Illustration of τ_B and τ_V in the two-day case. The green span represents the time length of τ_B and the white span represents the time length of τ_V.

IV. Experiments and an Approximation Approach

This section discusses the simulation results and compares them with the proposed pricing integration equations before. In the end, an approximation method for obtaining the BV options with the maturity time three days or above will be given. The Simulation parameters are shown in Table 1.

Table 1: Table of simulation parameters

Initial Stock Price S_0	100	Strike Price K	99
Barrier Level L	99	Volatility σ	0.30
Risk-free Interest Rate r	0	Dividend Payment D	0
Simulation Time Length T	1 day	Market Open Time Length τ_B	15 hours
Market Closed Time Length τ_V	9 hours	Time Step dt	30 seconds

A. One-Day BV Option

Fig 3 illustrates a brief simulation plot of what the simulated price paths look like for the one-day BV option. When the market is open, if the path reaches the barrier level, it will be absorbed. However, after the market has been closed, the price path will not be absorbed even if it has touched the barrier level. However, if it turns out to be lower than or equal to the barrier level at the time that the market opens the next day, the corresponding price path will be knocked out.

The approach to obtain the option price via simulation is to calculate the expectation value of the payoff at the end of the option maturity time, given that the number of price paths for simulation is large enough. The number of stock price paths adopted is 20000 and the results are summarized in Table 2. The error percentage is 0.770%. Via the comparison between the simulated one-day BV option price and the theoretical vanilla and barrier option prices, it can assure that the simulation price is an acceptable value since it should be cheaper than the plain vanilla option price while more expensive than the plain down-and-out barrier option price. The error percentage can be further minimized if a much smaller time step is adopted.

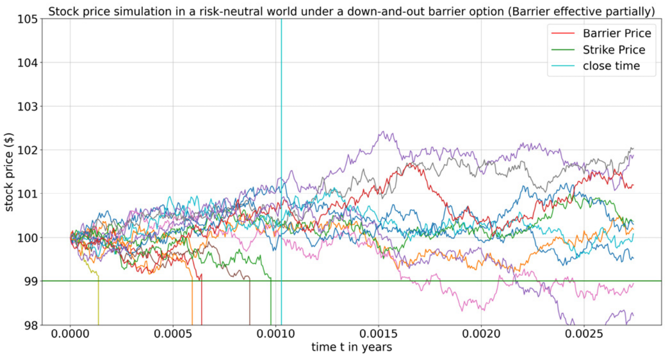

Fig. 3. Simulation stock price paths of the one-day BV option. S_0=100, K=B=99 and the number of paths is 15.

Table 2. Table of the prices of an one-day BV option via simulation and the integration formula. Both values are corrected to 7 significant figures. The number of stock price paths in the simulation is 20000

Simulation Results	Proposed Integration Formula
1.090705	1.082305
Theoretical Vanilla Option Price	Theoretical Down-and-out Barrier Option Price
1.246788	1.000000

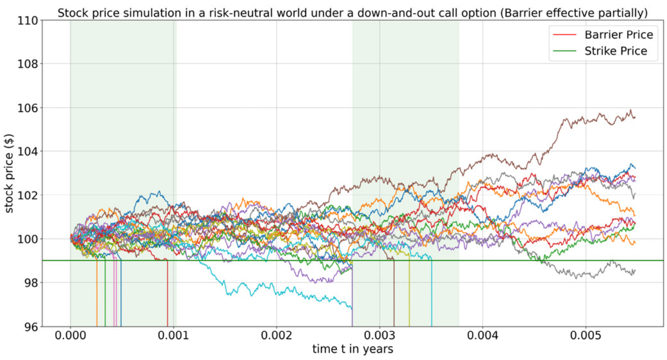

Fig. 4. Simulation stock price paths of the two-day BV option. S_0 = 100, K = B = 99 and the number of paths is 25.

B. Two-day BV Option

Fig. 4 illustrates the simulation plots of the stock price paths in the case of a two-day BV option. In addition to the one-day BV option situation, when the market reopens the next day, if the starting stock price of a path is below the barrier level, the path is automatically knocked out.

Applying the same approach to the pricing of the two-day BV options, we can arrive at Table 3. The simulation result is an acceptable value since it is between the corresponding theoretical vanilla option price and the down-and-out barrier option price. The error percentage is 0.0858%.

Table 3. Table of the prices of a two-day BV option via simulation and the integration formula. Both values are corrected to 7 significant figures. The number of stock price paths in the simulation is 50000. The time step is modified to be 1 second

Simulation Results	Proposed Integration Formula
1.111675	1.1107215
Theoretical Vanilla Option Price	Theoretical Down-and-out Barrier Option Price
1.470243	1.000000

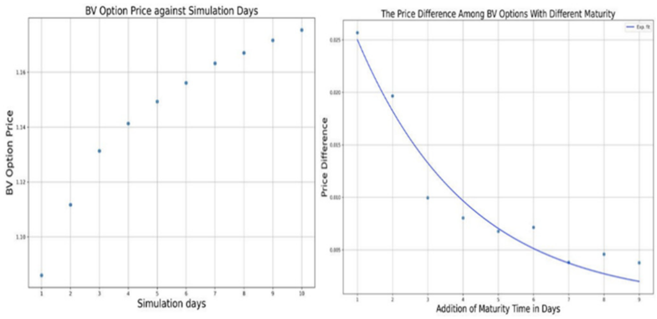

Fig. 5. The plot on the left illustrates the trend between the option prices and the N-day BV options, where N = 1, 2, …, 10. The plot on the right illustrates the option price changes as the number of simulation days increases. An optimized curve is fitted on the plot with the exponential model Ae^{Bt}, A, B $\in$ R.

C. An Approximation Method

As discussed previously, the option price can be obtained through the proposed method in utilizing integration formulas. However, as we proceed towards the BV options with more days, the integral dimension will increase up to N, which implies that it will be time- consuming if we insist on adopting this method for pricing BV options. The normal time to compute a two-day BV option is already more than one hour via Python and hence it will be much longer to compute for the BV options price with three days or above. Therefore, it is necessary for us to seek another way to compute the option price not necessarily accurately but approximately. The method is through a curve-fitting method.

Suppose we have already obtained the first few cases of BV options with N=1, 2, …, 10, we can calculate the difference in the option price among each consecutive case to attain the option price change when we continue to simulate for another additional day in the hope to look for patterns. The right plot in Fig. 5 shows such a fitted curve. The curve adopted the non-linear regression exponential model Ae^{Bt}, where $A, B \in R$. Through the curve-fitting optimization Python program, it is found that $A = 0.0343579 \pm 0.0032315$ and $B = -0.3169705 \pm 0.0367645$. The errors in the parameters are $\Delta A = 9.4\%$ and $\Delta B = 11.6\%$. Hence, with this curve, we can make approximations about the price of BV options with a long maturity time. For example, if we would like to calculate the price of a fifteen-day BV option, without consuming too much time on a Python program, we have $P_{15} = P_1 + \Sigma_{t=1}^{14} Ae^{Bt}$, where P_{15} denotes the fifteen-day BV option price and P_1 denotes the one-day BV option price.

V. Conclusion

An extension and modification of the approach in [4] is presented in this paper to give the pricing formula of the special down-and-out barrier option with the condition that the market closes on a non-dividend-paying stock. We proposed the pricing formula for the one-day BV option and extended it to the two-day case, from which we generalised it to the N-day case with the suggested approximation method to approximate the option price in the case of a very long maturity time. The approach in this paper is also valid for one-touch options and relevant variants such as other single-barrier options (down or up, call or put, in or out, for example). If the dividend payment is included, one can refer to [9] and modify Eq. (8) and Eq. (15) based on the approach proposed by [4] and thus obtain the pricing formulas with the conditions that the market closes and that the dividend payment exists. Moreover, this paper actually presents a pricing method of an integrated option type that consists of different fundamental options like Barrier and Plain-Vanilla options. Hence, one can also extend this reasoning to include more types of options to make the final pricing integral more general. Besides, in this study, we did not consider the effect of the presence of volatility on the special barrier option, which was assumed to be zero. However, one can refer to [10] for it. One of the potential limitations is the time it takes to obtain the results, since, as we increase the number of days, the dimensions of the pricing integral also increase. Hence, it will take much longer time to get the results and to solve this issue, an approximation method is proposed in the article.

Conflict of Interest

The author declares no conflict of interest.

References

[1] H. E. Gaarder, *The Complete Guide to Option Pricing Formulas*, New York, New York: McGraw-Hill, 1997.

[2] N. A. Chriss, *Black-Scholes and Beyond: Option Pricing Models*, McGraw-Hill: New York, 1997.

[3] J. Hull, *Options, Futures, and Other Derivatives*, New York, NY: Pearson, 2022.

[4] J. V. Siven, M. Suchanecki, and R. Poulsen, “Barrier options and lumpy dividends,” *Wilmott Journal*, vol. 1, no. 3, pp. 167–171, 2009.

[5] E. G. Haug, J. Haug, and A. Lewis, “Back to basics: A new approach to the discrete dividend problem,” *Wilmott Magazine*, pp. 37–47, 2003.

[6] H. Y. Wong and C. M. Chan, “Turbo warrants under stochastic volatility,” *Quantitative Finance*, vol. 8, pp. 739–751, 2008.

[7] J. M. Harrison and S. Pliska, “Martingales and stochastic integrals in the theory of continuous trading,” *Stochastic Processes and Their Applications*, vol. 11, pp. 215–260, 1981.

[8] Björk, *Arbitrage Theory in Continuous Time*, Oxford: University press, 2004.

[9] R. Bos, A. Gairat, and A. Shepeleva, “Dealing with discrete dividends,” *Risk*, pp. 111–112, 2003.

[10] B. Engelmann, M. Fengler, M. Nalholm, and P. Schwendner, “Static versus dynamic hedges: An empirical comparison for barrier options,” *Review of Derivatives Research*, vol. 9, pp. 239–264, 2007.

[11] S. E. Shreve, *Stochastic Calculus for Finance*, New York: Springer-Verlag, 2005.

[12] A. Etheridge, *A Course in Financial Calculus*, Cambridge: Cambridge Univ. Press, 2008.

[13] P. Wilmott, S. Howison, and J. Dewynne, *The Mathematics of Financial Derivatives: A Student Introduction*, Cambridge: Cambridge University Press, 2010.

[14] H. E. Gaarder, *The Complete Guide to Option Pricing Formulas*, New York, New York: McGraw-Hill, 1997.

[15] H. J. Michael and D. M. Kreps, “Martingales and arbitrage in multiperiod securities markets,” *Journal of Economic Theory*, vol. 20, pp. 381–408, 1979.

[16] R. Beneder and T. Vorst, “Options on dividend paying stocks,” in *Proc. the International Conference on Mathematical Finance, World Scientific*, 2001, pp. 204–217.

[17] N. A. Chriss, *Black-Scholes and Beyond: Option Pricing Models*, McGraw-Hill: New York, 1997.

[18] R. Poulsen, “Barrier Options and their static hedges: Simple derivations and extensions,” *Quantitative Finance* vol. 6, pp. 327–335, 2006.

[19] Barrier Options - University of Oxford. (January 16, 2024). [Online]. Available: https://people.maths.ox.ac.uk/howison/barriers.pdf

[20] D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion*, Berlin: Springer, 1999.

[21] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, Amsterdam: North-Holland, 2011.