VaR Trend Analysis from Discretized and Continuous Approaches

M. Maïza, A. Clavel, N. Hatem, and C. Recommandé

Abstract—**Present paper objective is to analyze how trends can be detected and interpreted using a GARCH (1, 1) model. Through VAR calculation along 2 distinct ways, the whole concept is to find out if trends can be identified in the return series of an index. Then this information can be used with an estimated probability to complement the VAR in order to get better anticipation of possible losses in a stressed environment. In addition, back testing on CVaR reliability compared to the VaR will be run as well.**

Index Terms—**CVaR, estimated shortfall, GARCH, model comparison, portfolio management, trend analysis, VAR.**

I. Introduction

The GARCH model — for Generalized Autoregressive Conditional Heteroskedasticity – has been introduced and improved by Engle and Bollerslev [1]. Initially mono variable, the model was acceptable enough to compute data from a single asset or index alone. However when applied to full index or portfolio, by far the most common case for this kind of model, the results given by GARCH model can be questionable.

As it does not provide each individual return of the assets composing the index, the model cannot integrate a correlation between all their variations. Furthermore, the size of the index or portfolio can matter as well, because of possible correlation issue, and systemic risk gets higher.

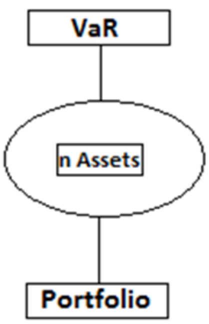

Fig. 1. VaR Calculation with Univariate GARCH model.

Either way GARCH model will provide a computed volatility which can be used to compute the Value at Risk (VaR) [2] Fig. 1 highlights that such a calculation with GARCH (1,1) model can be only run on the return of portfolio or index. For better index modeling [3], a calculation run with DCC–GARCH model can integrate return variations of different assets in the portfolio through their correlation matrix [4].

But instead of trying to use DCC–GARCH model approach [5], the study will focus on selecting indexes of different size, so that the model used on a small index could react as if it were in its multivariate form. Indeed in a small index, assets may be more correlated, so their variations will much more impact overall index return, which would not be as much noticeable with an index composed of hundreds of assets.

II. Problem Description

Since 2008, world economic crisis has raised the need of improved estimation of upcoming portfolio returns or index. Indeed, numerous models available to run simulations on a portfolio all have their plus and cons, and GARCH model is the most common one [6].

Though even in its DCC form, GARCH model provides better portfolio modelling [7]; better sensitivity to systemic risk may not be accurate enough in a stressful environment. During a crisis period where fluctuations are very large, volatility calculation might not be so accurate, and systemic risk, as it is a rather new aspect taken in consideration, might remain uncertain.

Thus major criticism regarding expected loss, calculated with VaR, is the specific issue of systemic risk estimation. Indeed while the economy runs smoothly, VaR calculation may provide accurate estimation of portfolio possible loss. But in crisis time, limits of any model are under question.

This is why, with so little back test, developing the approach of conditional VaR [8], aka CVaR calculation, is absolutely critical. Indeed with proper back testing on data of last decade, simulations can provide valuable results regarding CVaR accuracy to estimate upcoming shortfall of a portfolio or index during crisis time [9].

Furthermore, this will allow develop a new approach on GARCH model, which is to identify trends in CVaR, VaR and any model output. So in addition to taking into consideration the risk showed by VaR and CVaR aside, analysis of the trends could bring (if a variable probability) an additional measure of risk.

This new approach is known to have a real potential, because econometric models only have what can be called "horizontal" risk measure as they only use data at time t. Actually some can be used with a couple of periods, but it is a real difficulty to set the correct parameters. This is why trends, which will be identified on 3 periods, can bring a useful vertical approach to the model itself.

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III. BASE MODEL

A. Review Stage

To make accurate predictions related to possible loss of portfolio or index, and to test GARCH model efficiency, VaR and CVaR parameters computed with GARCH model will be first compared with the ones obtained by using raw data and basic formulae. As indicated earlier, the idea is to identify in a second time trends with a high probability of recurrent patterns. This can be adapted either to index returns, or to VaR and CVaR when the index is overwhelmed.

The 3 main equations of GARCH (1, 1) model [10] which is the backbone of present analysis are given by

$$r_t = \mu_t + a_t \tag{1}$$

$$a_t = h_t^{1/2} z_t \tag{2}$$

$$h_t = \alpha 0 + \alpha 1 a_{t-1}^2 + \beta 1 h_{t-1} \tag{3}$$

where $\alpha, \alpha 1$ and $\beta$are model parameters. In (1) r_t refers to the log return, μ_t to the expected value of r_t and a_t to the mean corrected return. In (2) h_t is the square of volatility and z_t are independent and identically distributed random variables.

Attention is focused on GARCH(1, 1) model as it is most reliable and easier to compute. Indeed as the final goal is to identify trends on a 3 periods interval, GARCH(3, 3) model would also provide outputs on the interval. However this later model is excessively difficult to work with as parameters α and β will be particularly hard to determine using common solvers.

CVaR calculation more specifically is based on the VaR here evaluated either directly from data or used with GARCH model in its parametric form. The parametric approach stands for a calculation with data variance and covariance. CVaR can be further developed, and its calculation implies a couple of steps. Actually first step is to get the measures of profit and loss of 2 portfolios or indexes. Then to obtain the joint law of these measures – $X(.)$ and $Y(.)$ here with x_n and y_n the *PnL* measures at day *n*:

$$\{p^{(XY)}(i,j) = \mathbb{P}[X(.) = x_i, Y(.) = y_j] \mid i \in \{1, \dots, I\}, j \in \{1, \dots, J\}\} \tag{4}$$

Next step consists in obtaining the performance of only one portfolio:

$$p_i^{(X)} = P[X(.) = x_i] = \sum_{j=1}^{J} p^{(XY)}(i,j) \tag{5}$$

Assuming that $x_m(x)$ and $y_m(y)$ are the respective values of greatest negative performance of each portfolio, the random variable loss_$X(.)$ takes its values in $\{O, \lambda_1^{(x)}, \dots, \lambda_i^{(x)}, \dots, \lambda_{m^{(x)}}^{(x)}\}$ and:

$$p^{(\text{loss\_}X)}(i) = p^{(X)}\left(m^{(x)} - (i-1)\right) \tag{6}$$

Assuming that $x_m(x)$ and $y_m(y)$ are the respective values of greater negative performance of each portfolio, loss_$X(.)$ takes its values in $\{O, \lambda_1^{(x)}, \dots, \lambda_i^{(x)}, \dots, \lambda_{m^{(x)}}^{(x)}\}$

$$p^{(\text{loss\_}X(.))}(0) \equiv P[loss_{X(.)} = 0] = \sum_{i=1}^{m^{(x)}} p^{(X)}(i) \tag{7}$$

$$p^{(\text{loss\_}X(.))}(i) \equiv P\left[\text{loss}_{X(.)} = \lambda_i^{(x)}\right] = P\left[X(.) = \right.$$ *xmx−i−1=pX(mx−i−1)* (8)

Fixing the system under normal market conditions, the event C_{norm}can be defined as:

$$C_{\text{norm}}(Y) = \{\mu_{(Y)} - \nu_{\sigma_{(Y)}} \leq \mu_{(Y)} + \nu_{\sigma_{(Y)}}\} \tag{9}$$

From the joint distribution, the limits of discretized intervals are

$$\begin{cases} \theta_{(Y)}^{-} = \mu_{(Y)} - \nu_{\sigma_{(Y)}} \\ \theta_{(Y)}^{+} = \mu_{(Y)} + \nu_{\sigma_{(Y)}} \end{cases} \tag{10}$$

Thus C_{norm} event probability can be reduced to the following simplified expressions:

$$P[C_{\text{norm}}(Y)] = \sum_{j=m_-^{(Y)}+1}^{m_+^{(Y)}} p^{(Y)}(j) \tag{11}$$

$$P[X(.) = x_i, C_{\text{norm}}(Y)] = \sum_{j=m_-^{(Y)}+1}^{m_+^{(Y)}} P[X(.) =$$ *xi,Y.=yj]=j=m−Y+1m+YpXYi,j* (12)

from which expected value X() can be calculated conditioned by the event $C_{\text{norm}}(Y)$:

$$\text{E}[X(.)|C_{\text{norm}}(Y)] = \sum_{i=1}^{I} x_i P[X(.) = x_i | C_{\text{norm}}(Y)] = (P[C_{\text{norm}}(Y)])^{-1} \sum_{i=1}^{I} x_i P[X(.) = x_i, C_{\text{norm}}(Y)] \tag{13}$$

On the other hand when considering a stressed market, the calculation is quite the same except for interval limits: lower limit is now defined by min($Y(.)$) and upper limit by VaR(N days, 99%).

	Performances									
	USA					G7				
	Intervalle	Occurrences	Proba	(Y-mu)^2	Pcum	Intervalle	Occurrences	Proba	(Y-mu)^2	Pcum
0	-190,92	1	0,43%	46091,66	100,00%	-188,03	1	0,43%	42574,57	100,00%
1	-158,64	0	0,00%	33273,66	99,57%	-159,71	0	0,00%	335,11	99,57%
2	-126,36	1	0,43%	22539,52	99,57%	-131,38	0	0,00%	335,11	99,57%
3	-94,08	5	2,13%	13889,26	99,15%	-103,06	2	0,85%	265,89	99,57%
4	-61,80	18	7,66%	7322,86	97,02%	-74,74	11	4,68%	53,38	98,72%
5	-29,53	19	8,09%	2840,32	89,36%	-46,41	12	5,11%	39,77	94,04%
6	2,75	49	20,85%	441,66	81,28%	-18,09	34	14,47%	246,30	88,94%
7	35,03	71	30,21%	126,86	60,43%	10,23	59	25,11%	1656,00	74,47%
8	67,31	53	22,55%	1895,93	30,21%	38,55	68	28,94%	2469,49	49,36%
9	99,59	16	6,81%	5748,87	7,66%	66,88	39	16,60%	428,24	20,43%
10	131,87	2	0,85%	11685,68	0,85%	95,20	9	3,83%	86,60	3,83%

Fig. 2. Filled intervals using discretized approach.

				[redacted]				
P[Cnorm(Y)]	90,21%	Co-Esp(0,99;X/beta;Y)	61,80		P[Cstress(Y)]	1,28%	Co-Esp(0,99;X/beta;Y)	190,92
Normal conditions		VaR	-61,80		Stress conditions		VaR	-190,92
Espérance	25,01				Espérance	-1,75		

Fig. 3. Results with normal and stressed market conditions.

Résumé PTF									
PTF	Infos PTF			Normal conditions			Stress conditions		
	Espérance	Variance	VaR	Espérance	Co-Expected shortfall	Co-VaR	Espérance	Co-Expected shortfall	Co-VaR
USA	7,62	2345,65	-116,47	3,08	42,02	-42,02	-1,93	190,92	-190,92
UK	6,22	3920,54	-169,78	3,83	76,06	-76,06	-2,29	218,37	-218,37
France	0,52	29,98	-14,68	0,37	10,33	-10,33	-0,17	14,68	-14,68
Canada	8,17	3019,82	-171,12	4,35	43,47	-43,47	-2,76	254,89	-254,89
Italy	0,23	21,84	-12,16	0,11	8,42	-8,42	-0,15	15,22	-15,22
Allemagne	0,55	33,13	-18,95	0,41	6,09	-6,09	-0,19	19,80	-19,80

Résumé Market								
Espérance	Variance	VaR	P[Cnorm]	P[Cstress]	Born Inf Cnorm	Born sup Cnorm	Born Inf Cstress	Born sup Cstress
5,50	165,42	-125,52	34,89%	1,28%	-9,41	20,42	-188,03	-125,52

Fig. 4. Overall VaR calculation with discretized approach.

Moments			Paramètres		
	Mean	0,005%		Alpha	6,33E-06
	Variance	0,000181757		Alpha1	0,200000443
	Skewness	0,018391196		Beta1	0,799999557
	Kurtosis	3,61		Conditions	1
	Standard Deviation Mean	1,36%			Alpha Alpha1 & Beta1 > 0
	Smoothing Parameter	0,94			

Calcul de Volatilité du CAC 40 en supposant que la Va

Close	Log Close	Return	Return ²	Conditional Variance	Likelihood	Conditional Standard Deviation
4013,97	8,30					
4012,91	8,30	-0,03%	0,000%	0,000181757	8,612469175	0,013481727
4017,67	8,30	0,12%	0,000%	=$H$2+$H$3*E12+$H$4	8,784031948	0,012318677
4024,80	8,30	0,18%	0,000%	0,000128011	8,938851945	0,011314204
4045,14	8,31	0,50%	0,003%	0,000109368	8,888468622	0,010457917
4043,09	8,30	-0,05%	0,000%	9,8907E-05	9,218756584	0,009945198

Fig. 5. Excel GARCH modeling with Parameters.

Calcul de VaR Paramétrique Normal Distribution

Confidance	95%	99%	99,9%
Significant	5%	1%	0,01%
Variable Aléatoire Associée	-1,64	-2,33	-3,72
VaR (1j)	**-2,21%**	**-3,13%**	**-5,01%**
Total Pertes Sup VaR	**73**	**22**	**4**
Probabilité Pertes Sup VaR	5,65%	1,70%	0,31%

Calcul de VaR Paramétrique Volat GARCH (1,1)

Confidance	95%	99%	99,9%
Significant	5%	1%	0,01%
Variable Aléatoire Associée	-1,64	-2,33	-3,72
VaR (1j)	**-2,23%**	**-3,15%**	**-5,04%**
Total Pertes Sup VaR	**70**	**22**	**4**
Probabilité Pertes Sup VaR	5,42%	1,70%	0,31%

Fig. 6. VaR Calculation with GARCH (1, 1)

IV. Simulations

The simulation process is based on a non-constant daily volatility computed with GARCH (1, 1) model [11]. Daily VaR can be computed with parametric formula using historical data over 5 years. Using discretized approach, VaR calculations are also run on a daily basis using the same data. They go through different steps, first one compute all asset returns. Next step is to identify all the intervals for the simulations and fill them with data; this is what Fig. 2 depicts with the intervals from 0 to 10 that are used for the computation, going along with the formulas.

Then C_{norm} and C_{stress} limits are determined from (10). After computing occurrence matrix, C_{norm} and C_{stress} are calculated.

Running the process with 6 out of the 7 indexes composing the MSCI World index, the following results are obtained, see Fig. 4, from market close data over 10 years period.

Before proceeding to trend analysis, it is interesting to note that, from these results, VaR is always located between Normal and Stressed CVaR. Now the focus will be set on simulation using GARCH model.

As noticeable on Fig. 3, GARCH model is used with Excel, so model parameters are obtained with its solver. Fig. 5 depicts another set of GARCH formulas and its parameters. On Fig. 6 there are the calculus run with the model showed in Fig. 5.

V. Trend Analysis

First trend analysis relies heavily on accuracy of previous calculations and efficiency of actually run back testing. Thus in order to start identifying trends and to look for patterns in obtained outputs, the first step is to create normal conditions for the test index in the same way as for the market. They can be categorized in four classes:

Once all different performances have been ranged using these 4 classes as showed in Fig. 7, trend detection starts by analyzing the data on performances at time t+1 and t+2 for each time t.

From Fig 8 and 9 one trend has been highlighted. When benefits are higher than market upper limit, index performance is situated in the normal market for the following 2 days. This means that if Class 4 is encountered at a time t, then Class 3 scenario will occur at times t+1 and t+2. This trend is identified here with a 100% precision.

In addition, with 71% precision, if performance is located in normal market at time t, it will remain in at times t+1 and t+2. In other words, there is 71% possibility to stay in Class 3 scenario 3 days in a row.

Class 1	Loss superior to the VaR
Class 2	Loss located between the VaR and the lower limit of the normal market
Class 3	Performance located in the normal market
Class 4	Benefits higher than the upper limit of the normal market

Fig. 7. Custom classes for trend analysis.

	Nb of trends				Proba of trends				
	1	2	3	4	1	2	3	4	
11	0	0	0	0	0,00%	0,00%	0,00%	0,00%	0
12	0	0	0	0	0,00%	0,00%	0,00%	0,00%	0
13	0	0	0	0	0,00%	0,00%	0,00%	0,00%	0
14	0	0	0	0	0,00%	0,00%	0,00%	0,00%	0
21	0	0	0	0	0,00%	0,00%	0,00%	0,00%	0
22	0	0	1	0	0,00%	0,00%	11,11%	0,00%	1
23	0	0	0	2	0,00%	0,00%	0,00%	11,76%	2
24	0	1	1	3	0,00%	12,50%	11,11%	17,65%	5
31	0	0	0	0	0,00%	0,00%	0,00%	0,00%	0
32	0	0	0	2	0,00%	0,00%	0,00%	11,76%	2
33	0	1	0	1	0,00%	12,50%	0,00%	5,88%	2
34	0	1	2	2	0,00%	12,50%	22,22%	11,76%	5
41	0	0	0	0	0,00%	0,00%	0,00%	0,00%	0
42	0	2	0	3	0,00%	25,00%	0,00%	17,65%	5
43	0	2	1	2	0,00%	25,00%	11,11%	11,76%	5
44	0	1	4	2	0,00%	12,50%	44,44%	11,76%	7
	0	8	9	17					

Fig. 8. Results of first trend analysis.

	Nb of trends				Proba of trends				
	1	2	3	4	1	2	3	4	
11	0	0	0	0	0,00%	0,00%	0,00%	0,00%	0
12	0	0	0	0	0,00%	0,00%	0,00%	0,00%	0
13	0	0	0	0	0,00%	0,00%	0,00%	0,00%	0
14	0	0	0	0	0,00%	0,00%	0,00%	0,00%	0
21	0	0	0	0	0,00%	0,00%	0,00%	0,00%	0
22	0	0	0	0	0,00%	0,00%	0,00%	0,00%	0
23	0	0	1	0	0,00%	0,00%	3,70%	0,00%	1
24	0	0	1	0	0,00%	0,00%	3,70%	0,00%	1
31	0	0	0	0	0,00%	0,00%	0,00%	0,00%	0
32	0	0	2	0	0,00%	0,00%	7,41%	0,00%	2
33	0	1	15	5	0,00%	50,00%	55,56%	100,00%	21
34	0	0	4	0	0,00%	0,00%	14,81%	0,00%	4
41	0	0	0	0	0,00%	0,00%	0,00%	0,00%	0
42	0	0	0	0	0,00%	0,00%	0,00%	0,00%	0
43	0	1	4	0	0,00%	50,00%	14,81%	0,00%	5
44	0	0	0	0	0,00%	0,00%	0,00%	0,00%	0
	0	2	27	5					

Fig. 9. Results of second trend analysis.

VI. Conclusion

To strengthen prediction capability of losses for portfolios and indexes, an approach based on trend detection resulting from improved GARCH model has been proposed. The method is using finer evaluation of VaR and CVaR in different market conditions. It has been possible here to identify one trend with very high probability, though more data and more calculations are needed to ascertain exact result reliability. This would allow extend the analysis and not just work on a reduced sample of the market. Actually this is the main important issue, because identification, when adding this new set of predictions tools, is the first step of a process which represents a useful complement to actual econometric models relying only on reaction,.

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