A Study on Analytical and Numerical Solutions of Three Types of Black-Scholes Models

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***Abstract*—In the history of option pricing, one of the most significant models is the Black-Scholes model. In this paper, the classical Black-Scholes model for European and American options and its two modified forms will be described, which are nonlinear Black-Scholes model and time-fractional Black-Scholes model. The main purpose of this paper is to discuss those most popular analytical and numerical solutions for solving each type of Black-Scholes model. Among those solutions, common schemes will be summarized and the qualitative analysis of those methods, such as stability and convergence, will be presented.**

***Index Terms*—Analytical solution, Black-Scholes model, nonlinear Black-Scholes model, numerical solution, time-fractional Black-Scholes model.**

I. INTRODUCTION

Option pricing becomes an amplified area of discussion and important problem for both financial and mathematical point of view during the last decades. An option is a bond between two parties in which the option buyer buys the right instead of the obligation to buy or sell an underlying asset at a pre-fixed strike price from or to the option writer within a fixed period. The options can be classified into two types, the call options and the put options, based on the option rights. The call option can bring the owner the right to buy at a specific price and the put option can bring the right of the owner to sell at a specific price. The options can also be classified into another two types, the American options and the European options, according to the options styles. European options can only be exercised on the expiration day while American option can be exercised at any time up to and including the expiry.

There are many models for valuation of options but among all of those models, the Black-Scholes model is a more appropriate way to calculate the European options price. In 1937, the Black-Scholes model was first disclosed in the paper and then advanced by Robert Merton. Chawla, approximating European put option value by generalized trapezoidal formula. Later, Crank-Nicolson method has been applied for valuation of European options with accuracy up to three decimal places. Black-Scholes formula was used to calculate the price of European call option. Monte Carlo method was used to value options with accuracy and reliability. Recently, Fourier series with Legendre polynomials was applied to price European option [1].

In this paper, three types of Black-Scholes models will be described in Section II. The focus of this paper is to discuss the methods for solving each type of Black-Scholes models under certain circumstances, which will be summarized in Section III. Section IV contains conclusion.

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II. CLASSICAL BLACK-SCHOLES MODELS AND ITS MODIFICATION

The theoretical value of options can be determined by the celebrated Black and Scholes option pricing formula, which was developed in 1973 by Fischer Black, Robert Merton and Myron Scholes and was based on certain parameters having six assumptions. The change in stock price of the underlying asset satisfies the stochastic differential equation $dS = (\mu - D)Sdt + \sigma SdW$ of the geometric Brownian motion with a drift $\mu - D$, where μ is the drift rate, D is the dividend yield, σ is the volatility of the relative price change of the underlying stock price and dW is the Wiener process. Using Ito's lemma and eliminating the market randomness, one can derive the well-known Black-Scholes partial differential equation as:

$$\boldsymbol{\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D)S\frac{\partial V}{\partial S} - rV = 0}, \quad (1)$$
$$S \in (0, \infty), t \in (0, T)$$
$$V(S, T) = \max(S - E, 0),$$

where S is the stock price, E is the strike price, r is the risk-free interest rate, T is the time to expiration in years.

The classical Black-Scholes model is a benchmark for option pricing. Several restrictive assumptions as expressed by the model to achieve a simple and closed form pricing formula are the following: 1) Based on European options, 2) The risk-free rate of interest is constant for different maturities, 3) The underlying assets follows a log-normally distribution with constant mean and variance rates, 4) It pays no dividends during the life of the options, 5) There is no transaction costs and no taxes, 6) There are no risk-less arbitrage opportunities, 7) Trading takes place continuously and the standard form of the capital market model holds at each instant.

However, there are some limitations of the standard Black-Scholes model. First, the volatility smile contradicts with the constant volatility assumption. Second, some stylized features, asymmetry, fat tails and high peaks reflect shows that the log return of stock price does not follow the normal distribution. Also, the leverage effect and the clustering effect in the market are not being considered in the model. The classical Black-Scholes formulas only correctly predict European options for the reason that European options hold an exact expiration data and it considers a geometric Brownian motion where the market might deviate from the original assumptions. Thus, more and more modified Black and Scholes models are proposed in order to relax some assumptions to better capture the market behaviors and to obtain more accurate option prices.

A. Nonlinear Black-Scholes Models

The first type of modified Black-Scholes model is nonlinear Black-Scholes model. In the last decades, a nonlinear modification of the classical Black-Scholes model takes nonlinear effects in the real market into consideration to relax some of the restrictive assumptions, such as market illiquidity and feedback effects because of the impact of large traders choosing given stock trading strategies (Frey and Patie [2]), risk from unprotected portfolio (Kratka [3]), imperfect replication and investors' preferences (Barles and Soner [4]), the presence of transaction costs (Sevcovic and Zitnanska [5]), etc.

In recent literatures, Edeki, Owoloko and Ugbebor [6] considered the non-constant volatility of the stock price and presented a modified model via the implement of the constant elasticity of variance (CEV) model, which is given by Cox and Ross (1976) as:

$$dS_t = S_t \mu dt + \sigma S_t^{\xi/2} d(W_t), \quad (2)$$

where ξ represents the elasticity rate, and W_t is a standard Brownian motion. By using the CEV model, the variations of underlying asset are negatively correlated with variations in the volatility level, which can reduce the volatility smile effects [6]. Zhu and He [7] combining the truncated normal distribution with martingale restriction which needs to be imposed to avoid arbitrage opportunities. The modified model was based on revised assumptions that the range of the underlying asset varies within a finite zone instead of varying in a semi-infinite zone and the log-returns of underlying assets follow a truncated normal distribution within this price range. Grossinho, Kord and Sevcovic [8] proposed a modified model for pricing American-style call options under variable transaction costs, transforming the free boundary problem into the Gamma variational inequality and set the volatility term depend on both the underlying asset price and the Gamma of the option. The extended version of the Black-Scholes equation is given as:

$$\partial_t V + (r - D)S\partial_S V + \frac{1}{2}\sigma^2 S^2 \partial_S^2 V - rV = r_{TC} S. \quad (3)$$

The transaction cost measure r_{TC} is given by $r_{TC} = \frac{E[\Delta TC]}{S\Delta t}$, where ΔTC is the change in transaction costs during the time interval ΔT. Based on the fact that trading does not take place instantaneously and simultaneously in the option and the underlying asset when applying the hedging strategy, Bellalah [9] modified the original assumption to account for the 'lag', which stands for the elapsed time between buying or selling the option and the underlying assets. Some other literature also took non-constant volatility and occurrence of bankruptcy into consideration based on the efficient market hypothesis, or developed an approach by following the American option trading method and modified the parameters strike price, time of maturity and volatility in original Black-Scholes model in order to calculating stock price in frontier markets.

B. Time-Fractional Black-Scholes Models

The second type of modified Black-Scholes model is time-fractional Black-Scholes model. The main advantage of fractional derivatives compared to classical integer-order models is that a fractional differential equation can provide an instrument for defining memory and hereditary properties of various materials and process. Some scholars use a fractional stochastic dynamics of stock exchange to derive the Fractional Black-Scholes model. The fractional Black-Scholes models are derived by substituting the standard Brownian motion with fractional Brownian motion. It has three advantages: the fractional derivative is a generalization of the ordinary derivative; the fractional derivative is a non-local operator; the fractional Black-Scholes model is more accurate.

Wyss [10] presented a time-tractional Black-Scholes model to price a European call option. Space fractional order Black-Scholes models for exotic options in markets with jumps was derived. By using the fractional order Taylor's formula and Ito's lemma, Jumarie [11] derived a time and space fractional Black-Scholes model for stock exchange dynamics,

$$\frac{\partial^\alpha P}{\partial t^\alpha} = \left(rP - rS\frac{\partial P}{\partial S}\right)\frac{t^{1-\alpha}}{(1-\alpha)!} - \frac{\alpha!}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} \quad (4)$$
$$P(S,T) = \max(E - S, 0),$$

where σ, r and E denote the volatility of the underlying asset, the risk-free interest rate and the strike price of the option respectively. The stock exchange dynamics is assumed to follow a stochastic differential equation

$$dS = rSdt + \sigma S b(t,\alpha) = rSdt + \sigma S\omega(t)(dt)^{\frac{\alpha}{2}}, \quad (5)$$

where $\omega(t)$ is a normalized Gaussian white noise with zero mean and the unit variance. A time fractional Black-Scholes-Merton model considering the connection of the fractal structure and an options diffusion process was proposed in 2009. Then, a single parameter and a bi-parameter fractional Black-Scholes model was developed assuming the underlying stock price follows a fractional Ito process. Kalantari and Shahmorad [12] used a fractional stochastic dynamics of stock exchange to obtain the Fractional Black-Scholes model and applied the quasi-stationary method to remove the free boundary problem in the Fractional Black-Scholes model proposed by Jumarie [11].

III. Analytical and Numerical Solution of Black-Scholes Equation

In this section, we will discuss the analytical and numerical solutions of different type of Black-Scholes equation in details. In most cases, the analytical solution of partial differential equation is very different to obtain and in a very complex form. Therefore, it is necessary to solve the partial differential equations numerically.

A. The Classical Black-Scholes Equation

For the analytical solution, when σ, r and D are constants, the classical Black-Scholes equation can be easily solved to obtain the closed form solution as follows:

$$C(S,t) = Se^{-DT}N(d_1) - Ee^{-rT}N(d_2)$$
$$P(S,t) = Ee^{-rT}N(-d_2) - Se^{-DT}N(-d_1), \quad (6)$$

where $d_1 = \frac{\ln\left(\frac{S}{E}\right)+\left(r-D+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$, $d_2 = \frac{\ln\left(\frac{S}{E}\right)+\left(r-E-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$, $N(d)$ is the value of the cumulative standardized normal distribution evaluated at d_1 and d_2. The Black-Scholes equation can also be solved directly using the Mellin transformation, which does not require either variable transformations or solving diffusion equations.

For the numerical solution, the first numerical approach was lattice technique and was improved, which was equivalent to an explicit time-stepping scheme. Other numerical method based on classical finite difference methods which was used to be applied to the constant coefficient heat equation have been developed.

Finite difference method is now one of the most popular methods to solve partial differential equations, which can be used to solve Black-Scholes equation with boundary condition. Dura and Mosneagu [13] applied finite difference methods on Black-Scholes model and obtained an explicit and two implicit methods (fully implicit and semi-implicit method) to solve the discretized system. Anwar and Andallah [14] used a finite difference scheme to approximate the solution numerically of Black-Sholes model for a European call option with final and boundary condition. They established the stability condition of the scheme through convex combination and calculated the estimation of relative error in L1-norm to test the accuracy. Hossan *et al*. [1] found the approximate solutions of the Black-Scholes model by two numerical techniques, Du Fort-Frankel finite difference method (DF&DM) and Galerkin weighted residual method (GWRM) for European options. It has proved that these methods can give more accurate results than the results obtained by the adomain decomposition method.

B. Nonlinear Black-Scholes Model

Efficient techniques and fast computational methods for pricing option is a practical task in financial market. Therefore, nonlinear generalizations of the Black-Scholes models, which takes realistic market effects into account have to be solved in an efficient and accurate way. However, there is no explicit formula except under special circumstances with non-standard pay-off diagrams. This is the reason why numerical solutions of nonlinear Black-Scholes models have to be developed and analyzed.

To obtain the numerical solution of nonlinear Black-Scholes model, some numerical schemes, such as finite difference, local/global Radial Basis Function (RBF) meshfree methods , six order finite difference (FD6) scheme in space and third-order strong stability preserving Runge-Gutta (SSP-RK3) in time (Gulen, Popescu and Sari [15]), Crank-Nicolson scheme (Ankudiova and Ehrhardt [16]), Newton linearization technique and the asymptotic perturbation method (Duris *et al*. [17]), adomain decomposition method can be applied.

Pricing European-Style option under the transaction costs, which are modeled by nonlinear PDES, is important for the reason that the transaction costs reduce the expected return and thus cannot be ignored. Leland [18] obtained the option price with the volatility $\tilde{\sigma} = {\sigma_0}^2(1 + Le \times sign(V_{SS}))$, where Le is the Leland number given by $Le = \sqrt{2/\pi}(k/(\sigma_0\sqrt{\delta t})$, δt and k represent the transaction frequency and transaction cost measure respectively. Then, another form of the volatility of the option price has been proposed: $\tilde{\sigma} = \sigma_0(1 + cA)^{1/2}, A = \mu/(\sigma_0\sqrt{\Delta T}), c = 1$, where μ is the proportional transaction cost, ΔT is the transaction period and σ_0 is the original volatility constant. Barles and Soner [4] proposed a nonlinear Black-Scholes equation with transaction costs and the implied volatility of the following form: $\sigma^2 = \sigma_0^2(1 + Z[\exp(r(T - t))\, a^2S^2V_{SS}])$, wherein the maturity time is T, $a = \mu\sqrt{\gamma \mathrm{N}}$, with factor of risk of aversion γ and N number of options for solding. Gulen, Popescu and Sari [15] combined a sixth order finite difference (FD6) scheme in space and a third-order strong stability preserving Runge-Gutta (SSP-RK3) in time to solve the nonlinear Black-Scholes model with the nonlinear volatility proposed by Barles and Soner [4]. Ghanadian and Lotfi [19] applied the same improved implied volatility into nonlinear Black-Scholes equation. The equation under transaction costs solved in Ghanadian and Lotfi [19] is given by:

$$V = \frac{1}{2}\sigma_0^2\big(1 + Z\big[\exp\big(r(T-t)\big)\, a^2S^2V_{SS}\big]\big)S^2V_{SS} + (r-q)SV_S - rV,$$
$$S > 0\ t \in [0,T). \quad (7)$$

They converted the equation to a set of ordinary differential equations and applied a new fourth-order finite difference approximation along with the fourth-order Runge-Kutta scheme to obtain the approximate solution, which was proved be of fourth-order accuracy in both time and space.

Volatility is one of the most considerable factors in the model. Various models relevant with the Black-Scholes equations with volatility depending on several factors, such as stock price, the time, the option and its derivatives due to transaction costs, has be proposed by Ankudiova and Ehrhardt [16]. They solved those nonlinear Black-Scholes equations numerically. The nonlinear Black-Scholes equations with a constant drift μ and a nonconstant volatility $\tilde{\sigma}^2 \coloneqq \tilde{\sigma}^2(t, S, V_S, V_{SS})$. Under these circumstances, the classical Black-Scholes becomes the following nonlinear Black-Scholes equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\tilde{\sigma}^2(t,S,V_S,V_{SS})S^2\frac{\partial^2 V}{\partial S^2} + (r-D)S\frac{\partial V}{\partial S} - rV = 0,$$
$$S \in (0,\infty), t \in (0,T), \quad (8)$$

Ankudinova and Ehrhardt [16] applied three modified volatility into the above nonlinear Black-Scholes model including transaction costs, the Leland's model, the Barles' and Soner's model and the Risk adjusted pricing methodology model. They used two finite difference schemes, Crank-Nicolson method and Rigal compact schemes for the numerical computation of the option prices. Duris *et al*. [17] focused on two classes of nonlinear Black-Scholes equation. One nonlinear volatility model was developed by Frey and Patie [11]. The nonlinear volatility function σ in the model is given by $\sigma(\partial_S^2 V, S) = \tilde{\sigma}(1 - \rho S \partial_S^2 V)^{-1}$, where $\tilde{\sigma}$ is a constant historical volatility. Another nonlinear model was proposed by Kratka [3]. They applied two numerical approximation methods based on the asymptotic perturbation analysis and the Newton linearization technique to solve a wide class of nonlinear Black-Scholes equations with the nonlinear volatility functions taking the following form:

$$\sigma(\partial_S^2 V, S, T-t)^2 = \tilde{\sigma}^2 + 2\varepsilon A(T-t)S^{\gamma-1}H^{\delta-1}, \qquad (9)$$

where $H = S\partial^2 V/\partial S^2$.

When including the dividend payments, the Black-Scholes model for pricing stock options is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \big(rS - D(S,t)\big)\frac{\partial V}{\partial S} - rV = 0, \qquad (10)$$
$$S \in (0,\infty), t \in (0,T)$$

Company, Gonzalez and Jodar [20] set $D(S,t) = A\delta(t - t_d)S$, where A is a constant, t_d is dividend date, $A\delta(t - t_d)$ is the shifted Dirac delta function. They derived the numerical solution of Black-Scholes equation pricing stock options with discrete dividend by using a delta-defining sequence of the involved generalized Dirac delta function and connected the Mellin transformation to get the integral formula. Then, the numerical solution was approximated by utilizing numerical quadrature approximations.

C. Generalized Black-Scholes Equation

In the real financial market, those parameters σ, r and D depend highly on stock price and time variable. In this case, the formula of generalized Black-Scholes model is as follows:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2(S,t)S^2\frac{\partial^2 V}{\partial S^2} + \big(r(S,t) - D(S,t)\big)S\frac{\partial V}{\partial S} - r(S,t)V = 0,$$
$$S \in (0,\infty), t \in (0,T), \qquad (11)$$

However, the closed form solution of generalized Black-Scholes model is often not available, many methods were applied to solve its solution analytically and numerically.

The existence and uniqueness of analytical solution of generalized Black-Scholes model can be found in Friedman [21], Isakov [22]. However, finding the solution of Black-Scholes equation is not easy when its coefficients are discontinuous ordinary functions or generalized functions. Those equation cannot be transformed into the standard heat equation and thus we cannot easily find its analytical solution. We need to use some other efficient and accurate numerical algorithms to obtain the approximate solution.

For the numerical solution of generalized Black-Scholes model, finite difference methods are numerical approach to approximate the option price, which can avoid probability theory and stochastic methods and resolves the degeneracy effectively. A fitted finite volume method for spatial discretization of the Black-Scholes equation was proposed. However, the method is only first-order convergent. Then, A numerical method was presented based on a central difference spatial discretization on piecewise uniform mesh and an implicit time stepping technique. The method is second-order convergent with respect to the space variable and is robust, but is only first-order convergent with respect to the time variable. Later, a fourth order compact finite difference scheme in space was constructed for modeling European options. Recently, Sari and Balacescu [23] proposed a fourth-order difference and MacCormack schemes for Black-Scholes option model. Awasthi and TK [24] applied generalized trapezoidal formulas (GTF) as a numerical time integration in generalized Black-Scholes model for the temporal discretization along with classical finite difference schemes in space direction to obtain an accurate solution.

Cubic B-spline collocation method have second-order accuracy in approximating the generalized Black-Scholes model. Kadalbajoo *et al.* [25] applied cubic B-Spline collocation method for solving the equation which is second-order convergence with respect to both variables, but is not robust. Later, they presented a cubic B-spline collocation method after transform the generalized Black-Scholes equation into non-degenerate uniformly parabolic PDE through logarithmic transform along with the Crank-Nicolsion time-stepping technique on nonuniform gird. The method is stable and second-order convergent for arbitrary volatility and interest rate. Huang and Cen [26] applied the implicit Euler method for time discretization and a cubic polynomial spline method for the spatial discretization, producing second-order convergence in spatial variable.

Other numerical methods can also be used for solving the generalized Black-Scholes equation, such as meshless approach, element-free kp-Ritz method, element-free Galerkin method, exponential time integration scheme, a simultaneous application of the high-order difference approximation with identity expansion (HODIE) scheme in the special direction and the two-step backward differentiation formula in the temporal direction [26], [27].

D. Time-Fractional Black-Scholes Model

The methods applied to obtain the analytical solution of the fractional Black-Scholes models are usually via integral transform methods, homotopy perturbation methods and homotopy analysis methods, wavelet based hybrid methods, separation of variables, Laplace homotopy perturbation method [28]. Prathumwan and Trachoo [29] investigated the dynamics of the option pricing through two-dimensional time fractional-order Black-Scholes equation for European put option. They used Liouville-Caputo derivative to improve the ordinary equation and applied Laplace homotopy perturbation method to obtain the analytic solution.

The analytical solutions are difficult to calculate for the reason that those methods take the form of convolution of some special functions or an infinite series with an integral. Thus, it is necessary and important to study the numerical approximate solutions of these models. For the numerical solutions, various effective methods have been used to solve the fractional Black-Scholes equation, For example, meshless local Petrov-Galerkin (MLPG), finite difference method, Grunwald-Letnikov approximation, adaptive moving mesh method, shifted Gruwald-Letnik scheme and backward difference techniques, predictor-corrector scheme based on the spectral-collocation method, multivariate Pade approximation method, implicit discrete scheme and θ finite difference scheme with first-order accurate in time and second order accurate in space, explicit-implicit numerical scheme with a low order of convergence, etc [29].

For the Wyss' time-fractional Black-Scholes equation,

$$\frac{\partial^\alpha C(S,t)}{\partial t^\alpha} + rS\frac{\partial C(S,t)}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C(S,t)}{\partial S^2} - rC(S,t) = 0, \quad (S,t) \in (0,\infty) \times (0,T), \tag{12}$$

with the terminal and boundary condition $C(0,t)) = p(t), C(\infty,t) = q(t), C(S,T) = v(S)$, where $0 < \alpha \leq 1$, T is the expiry time, r is the risk-free rate and $\alpha(\geq 0)$ is the volatility of the returns from the holding stock price S, $C(S,t)$ is the time-t price of a European double barrier option with underlying S. Zhang *et al.* [28] proposed implicit finite difference methods with a spatially second-order accuracy and temporally 2-α order accuracy and then the method was improved with the spatial accuracy to fourth-order. Golbabai and Nikan [30] presented a method based on the moving least-squares method to approximate the Wyss' time-fractional Black-Scholes equation.

For the Jumarie time-fractional Black-Scholes equation,

$$\frac{\partial^\alpha V}{\partial t^\alpha} = \left(rV - rS\frac{\partial V}{\partial S}\right)\frac{t^{1-\alpha}}{(1-\alpha)!} - \frac{\alpha!}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}, t > 0, \quad 0 < \alpha \leq 1, \tag{13}$$

where $V = V(S,t)$, r, α denote the risk-free rate and volatility respectively. The implicit finite difference scheme and the weighted finite difference scheme can be applied to obtain the numerical solution. Kalantari and Shahmorad [12] used a Grunwald-Letnikov scheme to solve the equation for pricing the American put option. Huang, Cen and Zhao [31] developed the adaptive moving mesh method which can capture the singular phenomena for Jumarie equation and a finite difference method to discretize the time-fractional Black-Scholes equation.

IV. Conclusion

In this article, the analytical and numerical methods to obtain the solution of the classical Black-Scholes model and two types of modified Black-Scholes models, the nonlinear Black-Scholes model and the time-fractional Black-Scholes model, are discussed. Finite difference method is now one of the most popular methods to solve partial differential equations, which can be used to solve Black-Scholes equation. For the classical Black-Scholes model, lattice technique and finite difference methods for numerical solution and Mellin transformation for analytical solution were proposed and developed. For time fractional Black-Scholes equation, there are many studies focused on Wyss' time-fractional Black-Scholes equation and Jumarie time-fractional Black-Scholes equation, which can both be solved by implicit finite difference scheme. For nonlinear Black-Scholes equation, more methods about numerical solutions were developed and analyzed for the reason that there is no explicit formula for nonlinear Black-Scholes equations except under special circumstances with non-standard pay-off diagrams. Many nonlinear Black-Scholes equations with non-constant volatility, transaction costs and discrete dividends were proposed and solved. A special type of nonlinear Black-Scholes equation, generalized Black-Scholes equation were developed, which can mainly be solved by two numerical methods, finite difference method and Cubic B-spline collocation method.

Conflict of Interest

The authors declare no conflict of interest.

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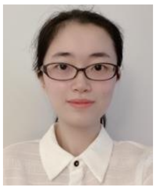

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