A Stochastic Model of the Variation of the Capital market Price

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***Abstract*— A stochastic model of the fluctuation of stock market price is considered herein. Precise conditions are obtained which determine the equilibrium price. Sufficient conditions for dynamic stability and convergence to equilibrium of the growth rate of the value function (out put) of stock shares are given. The model constrains the drift parameter of price process in such a way that it is fully characterized by the volatility.**

***Index Terms*—Bessel functions, Black- Scholes PDE. Stochastic model, Stock market Price variation.**

I. Introduction

Stock prices skyrocket with little reason, then plummet just as quickly (see figure 1 below), and people who have turned to investing for their children's education and their own retirement become frightened. Investors may 'temporarily' move financial prices away from their long term aggregate price 'trends'. Over-reactions may occur—so that excessive optimism may drive prices unduly high or excessive pessimism may drive prices unduly low. Economists continue to debate whether financial markets are 'generally' efficient. While efficient market hypothesis (EMH) predicts that all price movement is random (i.e., non-trending), many studies have shown a marked tendency for the stock market to trend over time periods of weeks or longer. Various explanations for such large and apparently non-random price movements have been promulgated. For instance, some research has shown that changes in estimated risk, and the use of certain strategies, such as stop-loss limits and **value at risk** limits, *theoretically could* cause financial markets to overreact (Jorion,1996 [10] and Singh,1997[15]). But the best explanation seems to be that the distribution of stock market prices is **non-Gaussian**.

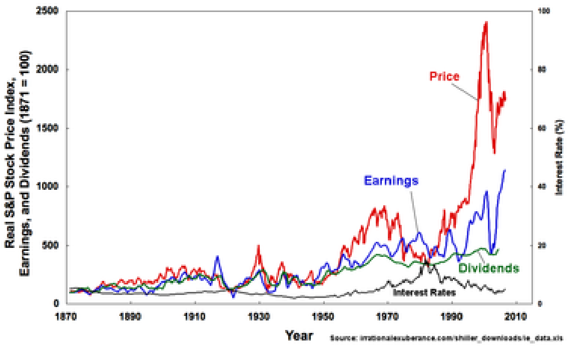

Figure1: Plot of the S&P Composite Real Price Index, Earnings, Dividends, and Interest Rates, from irrational Exuberance.

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Stock prices (viewed by some authors as a sequence of temporary equilibra, (Follmer,1994 [7]) , fluctuate widely in marked contrast to the stability of (government insured) bank deposits or bonds. This is something that could affect not only the individual investor or household, but also the economy on a large scale. The following deals with some of the risks of the financial sector in general and the stock market in particular. This is certainly more important now that so many newcomers have entered the stock market, or have acquired other 'risky' assets. The price evolution of a risky assets are usually modeled as the trajectory of a diffusion process defined on some underlying probability space, with the geometric Brownian motion the best candidate used as the canonical reference model. Brick (1987 [3]) had shown that geometric Brownian can indeed be justified as the rational expectations equilibrium in a market with homogeneous agents. Following Black and Scholes (1972 [1], 1973 [2]), a significant plateau has been reached by many authors in the model of stock price dynamics. Stein and Stein (1991 [16]), Heston (1993 [8]), Hull and White (1987 [9]) among others followed the traditional approach to pricing options on stocks with stochastic volatility which starts by specifying the joint process for the stock price and its volatility risk. Their models are typically calibrated to the prices of a few options or estimated from the time series of stock prices.

On the other hand, Ugbebor et al (2001 [14]) considered a stochastic model of price changes at the floor of stock market. Here the equilibrium price and the market growth rate of shares were determined. There have been some works with considerable extensions and constrains subsequently (see Osu and Okoroafor, 2007[11] and Osu et al, 2009 [12]). The aim of this paper is first; to present a dynamic stochastic model of variation of the capital market price aimed at determining the equilibrium price and growth rate of asset. The model constrains the drift such that it is characterized by the volatility. Hence the model assumes that stock price is a deterministic function of the stock price itself, so that the stock price is still the only source of uncertainty. Secondly, give sufficient conditions for stability and convergence to equilibrium. The remaining part of this paper is organized as follows. The next section is the set up. Section 3 is the model formulation; section 4 determines the equilibrium price. This paper ends with discussion and conclusion.

II. Mathematical Set Up

Let an investor observe prices and take actions in discrete time periods $t = 0,1,2, \dots$. The factors underlying price changes are uncertain and they are described in probabilistic terms. Uncertainty is modeled by a stochastic $x_t, t = 0, \pm 1, \pm 2, \ldots$, taking values in a measurable space X. The value of the random parameter x_t characterizes the "state of the world at time t (Evstigneev and Schenk-Hoppe, 2001[6]). Assume x_t follows the Orntsein-Uhlenbeck process,

$$dx_t = -ax_t dt + \sigma dB_t, \quad (1)$$

with explicit solution

$$x_t = e^{-at}x_0 + \sigma e^{-at}\int_0^t e^{-as}dB_s \quad (2)$$

(applying the Duhammel principle). Then (2) has a Gaussian distribution with mean $e^{-at}x_0$ and variance given by

$$\sigma^2(t) = \sigma^2 e^{-2at}\int_0^t e^{2as}ds$$
$$= \frac{\sigma^2 e^{-2at}}{2a}[e^{2as}+1]_0^t$$
$$= \frac{\sigma^2}{2a}[1+e^{-2at}]. \quad (3)$$

Hence (3) has a Markov process with stationary Gaussian transition probability densities

$$P(t,x,y) = \frac{1}{\sigma(t)\sqrt{2\pi}} exp\frac{\left[-(y - e^{-at}x)^2\right]}{2\sigma^2(t)}. \quad (4)$$

This is particularly interesting for $a > 0$ (say $a = 1$), which is the stable case and

$$\alpha = \lim_{t\to\infty}\sigma^2(t) = \frac{\sigma^2}{2} \quad (5)$$

and

$$\lim_{t\to\infty} P(t,x,y) = \frac{1}{\sqrt{2\alpha\pi}} exp\left(-\frac{y^2}{2\alpha}\right) \quad (6)$$

Thus as $t \to \infty$, $x_t \overset{d}{\to} N\left(0, \frac{\sigma^2}{2}\right)$.

Now consider a finite set of assets $i = 1, \ldots, n$ which can be any kind of financial assets, stocks being one of the most common example (amongst bonds and Options). The capacity out put (0r portfolio) is characterized by positions in these assets:

$$k = (k_1, \ldots, k_n).$$

Risk factors S describe any kind of risk and uncertainty present in the financial markets such as stock prices, interest rate etc.

$$S = (S_1, \ldots, S_m).$$

We denote by $V(S_t, k_t)$ the value of the capacity out put k for given values of risk factors S. Envision now an investor who starts with some initial endowment $v \geq 0$ and invests it in the $d+1$ assets described above. Let $N_i(k_t)$ denote the number of shares of asset i owned by the investor at time t. Then $V_0 = \sum_{i=0}^d N_i(0)s_i$ and the investor's capacity out put at time t is

$$V_t = \sum_{i=0}^d N_i(k_i)S_i. \quad (7)$$

If the trading of shares (and hence the adjustment of the portfolio) is allowed to take place only at discrete time points, say at $\ldots t-h, t, t+h \ldots$ and there is no infusion or withdrawal of fund, then

$$V_{t+h} - V_t = \sum_{i=0}^d N_i(k_i)[S_i(t+h) - S_i(t)]. \quad (8)$$

On the other hand, from time t to $t + \Delta t$, the dynamic growth of the portfolio is characterized by fluctuation due to some environmental effects (i.e., risk of stock price variation), such that $N = \frac{V_t(k_i, S_i)}{S_i}$, then (8) is now replaced by

$$V_{t+h} - V_t = \sum_{i=0}^d \left[V_t \frac{\{S_i(t+h) - S_i(t)\}}{S_i}\right] \quad (9)$$

The continuous-time analogue of (9) is (Osu,2010 [13])

$$dV_t = V_t(\alpha dt + \sigma dW_t). \quad (10)$$

III. Formulation

Given a finite time horizon $T > 0$, we shall consider a complete probability space $(\Omega, \mathcal{F}, P)$ equipped with a standard Brownian motion $W = (\{W_t^1, \ldots, W_t^d\}, 0 \leq t \leq T)$ valued in $\mathbb{r}^d$, and generating the (P-augmented) filtration $\mathcal{F}$.

The financial market consists of a non- risky asset S^0 normalized to unity, that is $S^0 = 1$, and d risky assets with price process $S = (S_t^1, \ldots, S_t^d)$ whose dynamics is defined by a stochastic differential equation (Etheridge, 2002 [5])

$$dS_t = \alpha S_t dt + \sigma S_t dW_t \quad (11)$$

It is not difficult to see (using Ito formula) that starting from S_0 at time 0, that the solution of (11) is

$$S_t = S_0 exp\left\{\sigma W_t + \left(\alpha - \frac{1}{2}\sigma^2\right)t\right\}, \forall\, t\varepsilon[0,T]. \quad (12)$$

Considering a short trading period where new dividends will not have been declared and no new assets have been purchased then the stock price follows the process

$$dS_t = \hat{\alpha} S_t dt + \sigma S_t dW_t, \hat{\alpha} = \alpha + \boldsymbol{\lambda}, \quad (13)$$

where λ is the market price of risk (Osu and Okoroafor, 2007 [11]). The stock pricing PDE is then the backward Black-Scholes PDE given (in one variable) as (Osu, et al, 2009 [12] and the references therein);

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial V}{\partial S^2} + \alpha S\frac{\partial V}{\partial S} - rV = -S \quad (14)$$

IV. Determination of Equilibrium Price Given Different Values of R

We now derive the equilibrium price given different values of r (the interest rate). Let $V(S)$ be twice continuously differentiable then (Osu, et al, 2009 [12]);

$$\frac{1}{2}\sigma^2 S^2 \frac{d^2V}{dS^2} + \alpha S\frac{dV}{dS} - rV = -S. \quad (15)$$

Case 1: $r = S + \alpha$

We assume r a linear function of price instead of a linear function of time as in Osu, et al, (2009 [12]). We now propose a solution of (15);

Proposition 1.

The solution of the time- homogeneous investment equation

$$\frac{1}{2}\sigma^2 S^2 \frac{d^2V}{dS^2} + \alpha S\frac{dV}{dS} - (\alpha + S)V = -S \quad (16)$$

where $r = \alpha + S$ is given as:

$$V = Ae^{\lambda_1 S} + Be^{\lambda_2 S} + 1. \quad (17)$$

Proof.

Let $\lambda_1$and λ_2 be the roots of the homogeneous part of (16), then

$$\left.\begin{aligned} \lambda_1 + \lambda_2 &= -\frac{2\alpha}{S\sigma^2} = -\frac{1}{S} \\ \lambda_1\lambda_2 &= \frac{-2(S+\alpha)}{\sigma^2 S^2} = -\frac{(S+\alpha)}{\alpha S^2} \end{aligned}\right\} \text{(using (5))} \quad (18)$$

We now have

$$\frac{d^2V}{dS^2} - (\lambda_1 + \lambda_2)\frac{dV}{dS} - \lambda_1\lambda_2 V = 0$$

or

$$\frac{d}{ds}\left(\frac{dV}{ds}-\lambda_2 V\right)=\lambda_1\left(\frac{dV}{ds}-\lambda_2 V\right) \quad (19)$$

Then

$$\frac{dV}{ds}=Z, \quad Z=\left(\frac{dV}{ds}-\lambda_2 V\right), \quad (20)$$

which gives $Z=ce^{\lambda_2 S}$ with solution

$$e^{-\lambda_1 S}V=\int c\, e^{(\lambda_1-\lambda_2)S}ds+B \quad (21)$$

(where c and B are arbitrary constants). Hence

$$V_c=Ae^{\lambda_1 S}+Be^{\lambda_2 S}, \quad (22)$$

where using (18) we have;

$$\left.\begin{array}{c}\lambda_1=\frac{1}{2}\left\{-1\pm\sqrt{1+4\left(\frac{S}{\alpha}+1\right)}\right\}\\ \lambda_2=-\left(\frac{1}{S}+\lambda_2\right)\end{array}\right\}, \quad (23)$$

and

$$A=\frac{c}{\lambda_1-\lambda_2}. \quad (24)$$

It is easy to see (using Euler's method on (14) and solving by variation of parameter) that

$$V_p=\frac{S}{r-\alpha} \quad (25)$$

We therefore replace r with $S+\alpha$ to get $V_p=1$, hence

$$V=V_c+V_p=Ae^{\lambda_1 S}+Be^{\lambda_2 S}+1 \quad .$$

as required.

The rate of growth of $V(S)$ is simply the rate of change in V expressed in relative (percentage) terms (Chiang and Wainwright, 2005 [4]), that is, expressed as a ratio to the value of V itself. Thus, for any given point of price S, we have

$$\text{Rate of growth of} \equiv \frac{dV/ds}{V}.$$

Since the growth rate of value function $V(S)$ is at the level of λ (λ_1,λ_2 are the copies of λ), our problem amounts to finding the value of S that maximizes V. The first-order condition for maximizing V is to have $\frac{dV}{ds}=0$. We obtain this by taken the natural log of both sides of (17) and then differentiating with respect to S, to get

$$InV(s)=InA+\lambda_1 S+InB+\lambda_2 S,$$

since $V\neq 0$, the condition $\frac{dV}{ds}=0$ can be satisfied if and only if

$$\frac{1}{V}\frac{dV}{ds}=\frac{1}{A}\frac{dA}{ds}=0,$$

using (23). Using (23) and (24),

$$\frac{dA}{ds}=A\left[-c\left\{\frac{-\frac{1}{S^2}+{}^{2}/_{\alpha}\left[1+4\left({}^{S}/_{\alpha}+1\right)\right]^{-1/2}}{\left[\frac{1}{S}-1+\left(1+4\left(\frac{S}{\alpha}+1\right)\right)^{1/2}\right]^2}\right\}\right]=0\,,$$

which gives

$$\frac{2S^2}{\alpha}\left[1+4\left({}^{S}/_{\alpha}+1\right)\right]^{-1/2}=1,$$

and

$$\hat{S}=\sqrt{\frac{\alpha}{2}},\ \ or\ \hat{S}=-\alpha. \quad (26)$$

Economically, only the first of (26) is admissible and so we rule out the negative prices. $\hat{S}$ is the equilibrium price since we have evaluated at the point where $\frac{dV}{ds}=0$. The next order of business is to verify if the value $\hat{S}$ meets the second-order condition of V. The second derivative of V is

$$\frac{d^2A}{dS^2}=A\frac{d}{dS}\left[-c\left\{\frac{-\frac{1}{S^2}+{}^{2}/_{\alpha}\left[1+4\left({}^{S}/_{\alpha}+1\right)\right]^{-1/2}}{\left[\frac{1}{S}-1+\left(1+4\left(\frac{S}{\alpha}+1\right)\right)^{1/2}\right]^2}\right\}\right]$$
$$+\left[-c\left\{\frac{-\frac{1}{S^2}+{}^{2}/_{\alpha}\left[1+4\left({}^{S}/_{\alpha}+1\right)\right]^{-1/2}}{\left[\frac{1}{S}-1+\left(1+4\left(\frac{S}{\alpha}+1\right)\right)^{1/2}\right]^2}\right\}\right]\frac{dA}{dS}.$$

But, since the final term drops out when we evaluate it at the equilibrium (optimal) point where $\frac{dV}{ds}=0$, we are left with

$$\frac{d^2A}{dS^2}=$$
$$-A\left[\left\{\frac{-\frac{1}{S^4}+2+\frac{2}{S^3}\left(\left[1+4\left({}^{S}/_{\alpha}+1\right)\right]^{-1/2}-1\right)+\frac{4}{\alpha^2}\left[1+4\left({}^{S}/_{\alpha}+1\right)\right]^{-3/2}\left(1-\frac{1}{S}\right)}{\left[\frac{1}{S}-1+\left(1+4\left(\frac{S}{\alpha}+1\right)\right)^{1/2}\right]^3}\right\}\right].$$

Discover that

$$\lim_{S\to\infty}\frac{d^2A}{dS^2}=\lim_{S\to\infty}\left[-A\left\{\frac{2+\frac{4}{\alpha^2}\left[1+4\left({}^{S}/_{\alpha}+1\right)\right]^{-3/2}}{\left[\frac{1}{S}-1+\left(1+4\left(\frac{S}{\alpha}+1\right)\right)^{1/2}\right]^3}\right\}\right]. \quad (27)$$

In view that $A>0$ (hence $V>0$), this second derivative is negative when evaluated at $\hat{S}>0$, thereby ensuring that the solution $\hat{S}$ is the indexed profit-maximizing.

Case 2

We now dare a solution of (14) when r is a quadratic function, that is $r=\alpha^2-S^2$. Replace in (14) to get;

$$\frac{1}{2}\sigma^2S^2\frac{d^2V}{dS^2}+\alpha S\frac{dV}{dS}+(S^2-\alpha^2)V=-S\,. \quad (28)$$

We write the homogenous part of (28) as,

$$S^2\frac{d^2V}{dS^2}+\frac{2\alpha S}{\sigma^2}\frac{dV}{dS}+\frac{2(S^2-\alpha^2)}{\sigma^2}V=0. \quad (29)$$

Let

$$u=\frac{\sqrt{2}}{\sigma}S\ \ \Rightarrow S=\frac{\sigma}{\sqrt{2}}u \quad (30a)$$

then

$$\frac{dV}{dS}=\frac{\sqrt{2}}{\sigma}\frac{du}{dS}\text{and}\frac{d^2V}{dS^2}=\left(\frac{\sqrt{2}}{\sigma}\right)^2\frac{d^2u}{dS^2}. \quad (30b)$$

Substituting this in (29) gives;

$$u^2\frac{d^2V}{du^2}+\frac{2\alpha u}{\sigma^2}\frac{dV}{du}+(u^2-\breve{\alpha}^2)V=0, \quad (31)$$

where $\breve{\alpha}=\frac{\alpha\sqrt{2}}{\sigma}$.

Using (5), (31) becomes;

$$u^2\frac{d^2V}{du^2}+u\frac{dV}{du}+(u^2-\alpha^2)V=0, \quad (32)$$

which is the Bessel's differential equation of order α, where α may be zero , integer and non-integer. It has solution of the first kind expressed as a series of gamma functions;

$$J_\alpha(u)=\frac{u^\alpha}{2^\alpha\Gamma(\alpha+1)}\left\{1-\frac{u^2}{2(2\alpha+1)}+\frac{u^4}{2.4(2\alpha+2)(2\alpha+4)}-\cdots\right\}$$
$$=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+\alpha+1)}\left(\frac{u}{2}\right)^{2m+\alpha}\ , \quad (33)$$

where $\Gamma(z)$ is the gamma function, a generalization of the factorial function to non-integer values and of the second kind of order α expressed in terms of the Bessel function of the first kind as;

$$Y_\alpha(u) = \frac{2}{\pi}J_\alpha(u)\left(In\frac{u}{2}+\gamma\right) - \frac{1}{\pi}\sum_{m=0}^{n-1}\frac{(n-m-1)}{m!}\left(\frac{u}{2}\right)^{2m-n}$$
$$+\frac{1}{\pi}\sum_{m=0}^{\infty}(-1)^{m-1}\left[\left(1+\frac{1}{2}+\cdots+\frac{1}{m}\right)+\left(1+\frac{1}{2}+\cdots+\frac{1}{m+n}\right)\right]\left(\frac{u}{2}\right)^{2m+\alpha}$$
$$= \frac{J_\alpha(u)\cos(\alpha\pi)-J_{-\alpha}(u)}{\sin(\alpha\pi)}, \qquad (34)$$

for non-integer α and $\gamma = 0.5772$ is the Mascheroni constant. Using (26) and (30a), we have $u = \sqrt{\frac{1}{2}}$. Clearly, $u < \sqrt{\alpha+1}$Therefore for small arguments
$0 < u \ll \sqrt{\alpha+1}$,

we obtain from (33) and (34)

$$J_\alpha(u) \to \frac{1}{\Gamma(\alpha+1)}\left(\frac{u}{2}\right)^\alpha, \qquad (35)$$

and

$$Y_\alpha(u) \to \begin{cases} \frac{2}{\pi}\left[In\left(\frac{u}{2}\right)+\gamma\right], & if\ \alpha = 0 \\ -\frac{\Gamma(\alpha)}{\pi}\left(\frac{2}{u}\right)^\alpha, & if\ \alpha \neq 0 \end{cases}. \qquad (36)$$

Equations (35) and (36) are the asymptotic forms of the Bessel functions for non-negative α. Thus the complementary solution of (31) ($\alpha \neq 0$) is now given as;

$$V_c = \frac{1}{\Gamma(\alpha+1)}\left(\frac{u}{2}\right)^\alpha - \frac{\Gamma(\alpha)}{\pi}\left(\frac{2}{u}\right)^\alpha$$
$$= \frac{1}{\alpha\pi(2u)^\alpha\Gamma(\alpha)}[\pi u^{2\alpha} - 2^\alpha\alpha\Gamma^2(\alpha)]. \qquad (37)$$

Using Euler's method on (14), solving by variation of parameter and replacing r with $\alpha^2 - S^2$ (using (26)) we obtain;

$$V_p = \frac{\sqrt{2\alpha}}{\alpha(2\alpha-3)}. \qquad (38)$$

Therefore,

$$V = \frac{1}{\alpha\pi(2u)^\alpha\Gamma(\alpha)}[\pi u^{2\alpha} - 2^\alpha\alpha\Gamma^2(\alpha)] + \frac{\sqrt{2\alpha}}{\alpha(2\alpha-3)}. \qquad (39)$$

On the hand, for large arguments $u \gg \left|\alpha^2 - \frac{1}{4}\right|$, (35) and (36) become;

$$J_\alpha(u) = \sqrt{\frac{2}{\pi u}}\cos\left(u - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) \qquad (40)$$

and

$$Y_\alpha(u) = \sqrt{\frac{2}{\pi u}}\sin\left(u - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right). \qquad (41)$$

We now have

$$V = \sqrt{\frac{2}{\pi u}}\cos\left(u - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) + \sqrt{\frac{2}{\pi u}}\sin\left(u - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) + \frac{\sqrt{2\alpha}}{\alpha(2\alpha-3)}, \qquad (42)$$

which becomes for $\alpha = \frac{1}{2}$

$$V = 2\sqrt{\frac{2}{\pi u}}\cos\left(u - \frac{\pi}{2}\right) + \frac{\sqrt{2\alpha}}{\alpha(2\alpha-3)}. \qquad (43)$$

This implies that, V_c is a circular function of u. Taking u alone as the independent variable, repetition will occur every time u increases by $\frac{5\pi}{2}$, since the expression $\cos\left(u - \frac{\pi}{2}\right)$ is a circular function of u, with period 2π and amplitude 1 (i.e. a circular function of u with period $\frac{5\pi}{2}$). Notice that $V \to V_p$ if and only if $\sqrt{\frac{2}{\pi u}} \to 0$ as $u \to \infty$ (i.e. a constant output growth of a firm or individual investment).

Case 3

We now consider a case where $r = S^2 + \alpha^2$, and we have (14) becomes;

$$\frac{1}{2}\sigma^2 S^2\frac{d^2V}{dS^2} + \alpha S\frac{dV}{dS} - (S^2+\alpha^2)V = -S, \qquad (44)$$

whose homogeneous part becomes (using (5) (30a,b))

$$u^2\frac{d^2V}{du^2} + u\frac{dV}{du} - (u^2+\alpha^2)V = 0. \qquad (45)$$

This is a modified Bessel differential equation of order α with two linearly independent solutions;

$$I_\alpha(u) = i^{-\alpha}J_\alpha(iu) = \sum_{m=0}^{\infty}\frac{1}{m!\Gamma(m+\alpha+1)}\left(\frac{u}{2}\right)^{2m+\alpha}, \qquad (46)$$

and

$$K_\alpha(u) = \frac{\pi}{2}\frac{I_{-\alpha}(u)-I_\alpha(u)}{\sin(\alpha\pi)}$$
$$= \frac{\pi}{2}i^{\alpha+1}H_\alpha^{(1)}(iu)$$
$$= -\frac{\pi}{2}i^{\alpha+1}H_\alpha^{(2)}(-iu), \qquad (47)$$

where

$$\begin{cases} H_\alpha^{(1)} = \frac{J_{-\alpha}(u) - e^{-\alpha\pi i}J_\alpha(u)}{i\sin(\alpha\pi)} \\ H_\alpha^{(1)} = \frac{J_{-\alpha}(u) - e^{-\alpha\pi i}J_\alpha(u)}{-i\sin(\alpha\pi)}, \end{cases} \qquad (48)$$

are the Hankel functions. If $u \gg \left|\alpha^2 - \frac{1}{4}\right|$, then (46) and (47) become;

$$I_\alpha(u) \approx \frac{e^u}{\sqrt{2\pi u}} \qquad (49)$$

and

$$K_\alpha(u) \approx \sqrt{\frac{\pi}{2u}}e^{-u} \qquad (50)$$

so that

$$V_c = \frac{1}{\sqrt{2\pi u}}[e^u + \pi e^{-u}]. \qquad (51)$$

Again, using Euler's method on (14), solving by variation of parameter and replacing r with $S^2 + \alpha^2$ (using (26)) we obtain;

$$V_p = \frac{\sqrt{2\alpha}}{\alpha(2\alpha-1)}. \qquad (52)$$

Therefore,

$$V = \frac{1}{\sqrt{2\pi u}}[e^u + \pi e^{-u}] + \frac{\sqrt{2\alpha}}{\alpha(2\alpha-1)}. \qquad (53)$$

Notice also that as $u \to \infty, V_c \to 0$ and $V \to V_p$. On the other hand for small arguments $0 < u \ll \sqrt{\alpha+1}$, (46) and (47) become;

$$I_\alpha(u) \approx \frac{1}{\Gamma(\alpha+1)}\left(\frac{u}{2}\right)^\alpha \quad (54)$$

and

$$K_\alpha(u) \approx \begin{cases} -In\left(\frac{u}{2}\right) - \gamma, if\ \alpha = 0 \\ \frac{\Gamma(\alpha)}{2}\left(\frac{2}{u}\right)^\alpha, \quad if\ \alpha > 0 \end{cases}. \quad (55)$$

These give for $\alpha > 0$;

$$V = \frac{1}{2^\alpha 2u^\alpha \alpha\Gamma(\alpha)}[\alpha 2^{2\alpha}\Gamma^2(\alpha) + 2u^{2\alpha}] + \frac{\sqrt{2\alpha}}{\alpha(2\alpha-1)}. \quad (56)$$

V. Determination Of The Volatility, Σ^2.

It is clear from equation (5) that the drift parameter α depends on the volatility σ. This constrains the drift parameter to be characterized by the volatility. Hence we assume that stock price is a deterministic function of the stock price itself, so that the stock price is still the only source of uncertainty.

$V(S)$ is an Ito process,therefore (see Ugbebor,2001 [14]);

$$dV = \left[\frac{dV}{dt} + \alpha S\frac{dV}{dS} + \frac{1}{2}\sigma^2 S^2 \frac{d^2V}{dS^2}\right]dt + \sigma S\frac{dV}{dS}dW.$$

The diffusion term is given by

$$\sigma S\frac{dV}{dS}dW = \sigma S\frac{dV/dS}{V}.VdW ,$$

with diffusion coefficient defined by

$$\sigma^2.(\overline{\Delta S})^2 = E[(\Delta S_i)^2] - [E(\Delta S_i)]^2.$$

So that

$$\sigma^2 = \frac{\{E[(\Delta S_i)^2] - [E(\Delta S_i)]^2\}}{(\overline{\Delta S})^2}. \quad (57)$$

VI. Discussion And Conclusion

The complementary function of (17) consists of two exponential expressions. The condition for dynamic stability of equilibrium depends on the algebraic signs of the characteristic roots. The coefficient A is a function of S, its value hinges on S and the initial conditions of the problem. B is an arbitrary constant whose value hinges on the initial conditions of the problem. We can be sure of a dynamically stable equilibrium ($V_c \to 0\ as\ S \to 0$), regardless of what the initial conditions happen to be, if and only if the roots λ_1 and λ_2 are both negative as the condition for dynamic stability does not permit even one of roots to be positive.

The V_c's represent the deviation from equilibrium and V_p's the intertemporal equilibrium level of the relevant variable. In the case of (17) the V_p is a constant, so we have a stationary equilibrium in the intertemporal sense

Notice that in (56) as $u \to \infty$, $\frac{1}{2^\alpha 2u^\alpha \alpha\Gamma(\alpha)} \to 0$, hence $V_c \to 0$and $V \to V_p$, also in (39) as $u \to \infty$, $\frac{1}{\alpha\pi(2u)^\alpha\Gamma(\alpha)} \to 0$ and $V \to V_p$. By (30a), u is a function of S and $u(S) \to \infty$ as $S \to 0$. In (39), $V_p < 0$ for $0 < \alpha \le 1$. Thus the value of output of individual investor (or portfolio of a firm) becomes negative. A crash has thus occurred in the capital market. A crash is a significant drop in the total value of a market, creating a situation wherein the majority of investors are trying to flee the market at the same time and consequently incurring massive losses. Attempting to avoid more losses, investors during a crash are panic selling, hoping to unload their declining stocks onto other investors. This panic selling contributes to the declining market, which eventually crashes and affects everyone. Crashes in the stock market is followed by depression, the values of shares held by investors are much less than their initial investment. For $\alpha \ge 2$, $V \to V_p > 0$. In (39), $V_p > 0$ for $\alpha \ge 1$. But for $\alpha = \frac{1}{2}$, then $V(S)$ increases without bound. Hence a bubble has just occurred, i.e. a sharp rise in value of an asset or a range of assets in a continuous process, with the initial rise generating expectations of further rises and attracting new buyers-generally speculators, interested in profits from trading in the asset rather than its use as earning capacity.

Note 1: This definition of a bubble is not interesting in a perfect foresight environment because it means that either the bubble goes on indefinitely, or if a crash is expected at some future date, the bubble will not start (because of backward induction). This has led to the efficient market view that bubbles cannot occur.

Note 2: If the intertemporal equilibrium is stationary then $V \to V_p$ if and only if $V_c \to 0$ as $S\ or\ u(S) \to 0$. This implies that V_p (and hence V) is dynamically stable. But if V_p is a moving equilibrium, then its plot is a curve rather than a horizontal straight line, hence stability is not attained. In fact this is the actual representation of the fluctuation of the capital market; a series of business (investment) cycles around a secular trend.

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